MATH2201. 2012. SOLUTIONS.

Answer ALL questions from Section A.

All questions from Section B may be attempted, but only marks obtained on the best **two** *solutions from Section B will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

Section A

1.

(a) Using the Euclidean algorithm calculate $gcd(60, 36)$.
ANSWER. One finds $gcd(60, 36) = 12$

(b) Find integers h and k such that

$$gcd(60, 36) = 60h + 36k.$$

ANSWER. $h = -1, k = 2$

(c) Does the equation $60x + 36y = 24$ have solutions (x, y) with x and y integers? If yes, find them all.
ANSWER. There are solutions, as 12 divides 24. The set of all solutions is $\{-2 + 3n, 4 - 5n : n \in \mathbb{Z}\}$.
Answer the same question with $60x + 36y = 6$.
ANSWER. No solutions, as 12 does not divide 6.

2.

(a) State the Chinese remainder theorem.
ANSWER. Let m and n be coprime integers. Let $x, y \in \mathbb{Z}$. There exists a unique $[z] \in \mathbb{Z}/mn\mathbb{Z}$ such that $z \equiv x \bmod m$ and $z \equiv y \bmod n$

(b) In each case below,does there there exist a $z \in \mathbb{Z}$ satisfying the given congruences? If yes, find one.

(i) $z \equiv 1 \bmod 2$ and $z \equiv 2 \bmod 4$
ANSWER. No solutions: the first equation implies that z is odd, the second, that z is even.

(ii) $z \equiv 2 \bmod 3$ and $z \equiv 3 \bmod 5$
ANSWER. Chinese remainder theorem applies. One can take for example $z = 8$.

(iii) $z \equiv 1 \bmod 2$ and $z \equiv 1 \bmod 4$

ANSWER.

Yes. For example $z = 5$.

3.

Consider the linear map represented by the following matrix in the standard basis of $\mathbb{C}^3$:

$$\begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Find the minimal polynomial, Jordan basis and Jordan normal form.

ANSWER.

5 is the only eigenvalue. $m_T(x) = (x-5)^2$.

Jordan basis: $\{(1,0,0)^t, (0,0,1)^t, (0,1,0)^t\}$.

Jordan normal form has one 2×2 block and one 1×1 block.

4.

Let T be a linear map such that $ch_T(x) = (x-2)^4(x-5)^3$ and $m_T(x) = (x-2)^2(x-5)^3$.

In each case below, find the Jordan normal form of T.

(i) $\dim(V_1(2)) = 2$

ANSWER. For eigenvalue 2, there are two 2×2 blocks.

For eigenvalue 5, there is one 3×3 block.

(ii) $\dim(V_1(2)) = 3$

ANSWER. For eigenvalue 2, there is one 2×2 block and two 1×1 blocks.

For eigenvalue 5, there is one 3×3 block

5.

Consider the following quadratic form:

$$q(x, y, z) = x^2 - z^2 + 2xy + 2yz$$

Find the real canonical form, rank and signature.

ANSWER.

One finds

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The signature is $(1, 1)$, rank is two.

Section B

6. (a)

Let k be a field and $f \in k[x]$ a non-zero polynomial. Show that $\lambda \in k$ is a root of f if and only if $x - \lambda$ divides f.

Deduce that f has at most $\deg(f)$ distinct roots.

ANSWER.

Suppose that $f(\lambda) = 0$. By Euclidean division,

$$f = (x - \lambda)q + r$$

with $deg(r) < 1$ hence r is a constant polynomial.

We also see that $r(\lambda) = 0$. But, r being constant, this implies that $r = 0$, hence $x - \lambda$ divides f.

The converse is trivial.

Let $\lambda_1, \ldots \lambda_r$ be distinct roots of f. Each $x - \lambda_i$ divides f and, as λ_is are distinct, they are coprime. This implies that $(x - \lambda_1)\cdots(x - \lambda_r)$ divides f. It follows that $r \leqslant deg(f)$.

(b) (i)

Say what is meant by an irreducible polynomial in $k[x]$. State the unique factorisation theorem for polynomials.

ANSWER. f is irreducible if whenever $f = gh$, g or h is a unit.

The unique factorisation theorem says that any f is a monic polynomial, there exist $p_1, \cdots, p_r$ monic irreducible such that $f = p_1 \cdots p_r$. This decomposition is unique: if $f = q_1 \cdots q_s$ with q_i monic irreducible, then $r = s$ and, after permutation, $q_i = p_i$ for all i.

Let f be a polynomial of degree 2 that has no roots in k. Show that f is irreducible.

ANSWER.

Suppose f is reducible $f = gh$. As $deg(f) = 2$, either g or h has degree one, hence has a root.

Let f be a polynomial of degree 3 that has no roots in k. Show that f is irreducible.

ANSWER.

Similarly, suppose f reducible, $f = gh$.Then $3 = deg(g) + deg(h)$. Hence one of them, say g, has degree one and therefore has a root.

Give an example of a field k and polynomial of degree 4 in $k[x]$ which has no roots in k but is not irreducible.

ANSWER.

In $\mathbb{R}[x]$, $x^4 + 1$ has no roots but $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.

(ii)

Let $f(x) = x^5 + x + 1$ in $\mathbb{F}_2[x]$. Using Euclidean division, show that $x^3 + x^2 + 1$ divides f and hence find a factorisation of f into irreducible factors in $\mathbb{F}_2[x]$. Justify why your factors are irreducible.

ANSWER.

By Euclidean division in $\mathbb{F}_2[x]$,

$$x^5 + x + 1 = (x^3 + x^2 + 1)(x^2 + x + 1)$$

This is a factorisation. The factors have degrees two and three respectively and no roots hence irreducible.

(iii)

Let $f \in \mathbb{F}_2[x]$ be a polynomial such that $d(x) = gcd(f, f + (x^2 - 1))$ is a polynomial of degree one. Find $d(x)$.

ANSWER.

d divides f and $f + (x^2 - 1)$. Hence d divides $x^2 - 1 = (x - 1)^2$. As d has degree one, it is $x - 1$.

7. (a)

Let V be a vector space over a field k of dimension n and $T\colon V \longrightarrow V$ be a linear map.

(i) Let $f \in k[x]$ be a non-zero polynomial such that $f(T) = 0$.
For each statement below, say whether it is TRUE or FALSE. Provide a proof or a counter-example accordingly.

(i) If $\lambda \in k$ is an eigenvalue of T, then $f(\lambda) = 0$.

ANSWER.

TRUE. We have $T(v) = \lambda v$ for some $v \neq 0$. Then $f(T)v = f(\lambda)v = 0$. As $v \neq 0$, $f(\lambda) = 0$.

(ii) If $\lambda \in k$ is a root of f, then λ is an eigenvalue of T.

ANSWER.

FALSE. Take T to be the identity and $f(x) = (x - 1)(x - 2)$. Then $f(T) = 0$, 2 is a root of f but not an eigenvalue of T.

(ii)

Let V be a vector space over a field k of dimension n. Let $T\colon V \longrightarrow V$ be a linear map.

Explain what is meant by the minimal polynomial m_T of T.

Let ch_T be the characteristic polynomial of T. Show that m_T divides ch_T.

Show that $\deg(m_T) \leqslant n$.

State the criterion for diagonalisability of T in terms of m_T.

ANSWER.

The minimal polynomial m_T is the unique monic polynomial such that $m_T(T) = 0$ and whenever $f(T) = 0$ for some non-zero f in $k[x]$, $\deg(m_T) \leqslant \deg(f)$.

Write $ch_T = qm_T + r$ with $\deg(r) < \deg(m_T)$. One has $r(T) = 0$ hence r is zero and m_T divides ch_T.
The characteristic polynomial ch_T has degree n and $ch_T(T) = 0$, hence $deg(m_T) \leqslant n$.
T is diagonalisable if and only if $m_T = (x - \lambda_1) \cdots (x - \lambda_r)$ where the λ_is are distinct eigenvalues.

Let $T\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a linear map. In each case below, give, with justification, the minimal polynomial of T and the Jordan normal form of T.

(b) (i) The eigenvalues of T are 1 and 2.
ANSWER.
T has two distinct eigenvalues hence diagonalisable. The minimal polynomial is $(x-1)(x-2)$ and the Jordan normal form is $diag(1,2)$.

(ii) $T^2 = 0$ and $T \neq 0$
ANSWER.
The minimal polynomial is x^2 and the Jordan normal form has one 2×2 block.

(iii) T is not invertible and 1 is an eigenvalue of T
ANSWER.
T is not invertible hence 0 is an eigenvalue. The other eigenvalue is 1. T has two distinct eigenvalues, hence diagonalisable. The minimal polynomial is $x(x-1)$ and the Jordan normal form is $diag(0,1)$.

8.

Let $(V, \langle , \rangle)$ be an inner product space.

(a.i) Prove that for any $x, y \in V$, one has

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

ANSWER.

$$< x+y, x+y > + < x-y, x-y > = < x,x > + < x,y > + < y,x > + < y,y >$$

$$+ < x,x > - < x,y > - < y.x > + < y,y >$$
$$= 2(||x||^2 + ||y||^2)$$

(a.ii) Explain what is meant for two vectors $x, y \in V$ to be orthogonal.
Show that if x and y are orthogonal, then

$$||x||^2 + ||y||^2 = ||x+y||^2$$

Show that the converse holds if $(V, \langle , \rangle)$ is a *real* inner product space.
ANSWER
x and y are orthogonal if $\langle x, y \rangle = 0$.

Since

$$||x+y||^2 = \langle x+y, x+y\rangle = ||x||^2 + 2\Re\langle x,y\rangle + ||y||^2,$$

so we have

$$||x||^2 + ||y||^2 = ||x+y||^2$$

if x and y are orthogonal.

When V is a real inner product space, the converse holds because then $\Re\langle x,y\rangle = \langle x,y\rangle$.

If $||x||^2+||y||^2 = ||x+y||^2$ then by the above calculation, $\Re\langle x,y\rangle = \langle x,y\rangle = 0$ hence x, y orthogonal.

(a.iii) Suppose that two vectors $x, y \in V$ satisfy

$$\langle x,z\rangle = \langle y,z\rangle$$

for any $z \in V$. Show that $x = y$. (Hint. Let $z = x - y$)

ANSWER

For all $z \in V$, $< x-y, z >= 0$. Let $z = x - y$, one finds that $||x-y|| = 0$, hence $x = y$.

(b)

Let $(V, \langle , \rangle)$ be an inner product space, let $T\colon V \longrightarrow V$ be a linear map and let T^* be the adjoint of T.

Suppose that $T^* = T$. Show that the eigenvalues of T are real.

Show that eigenvectors corresponding to two different eigenvalues are orthogonal.

ANSWER.

Let λ be an eigenvalue of T and let $v \neq 0$ be a corresponding eigenvector. Then

$$Tv = \lambda v$$

It follows that

$$< Tv, v >= \lambda < v, v >=< v, Tv >= \overline{\lambda} < v, v >$$

As $v \neq 0$, we can divide by $< v, v >= ||v||^2 \neq 0$. It follows that $\lambda = \overline{\lambda}$

Let now λ and μ be two distinct eigenvalues (we just saw that they are real). If $u \in V_1(\lambda)$ and $v \in V_1(\mu)$ then

$$\lambda\,\langle u,v\rangle = \langle \lambda u, v\rangle = \langle Tu, v\rangle = \langle u, T^*v\rangle = \langle u, Tv\rangle = \langle u, \mu v\rangle = \mu\,\langle u,v\rangle\,.$$

So $(\lambda-\mu)\,\langle u,v\rangle = 0$, with $\lambda \neq \mu$. Hence $\langle u,v\rangle = 0$.